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An axially symmetric scalar field and teleparallelism

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Abstract. An axially symmetric scalar field is considered in teleparallel gravity. We calculate, respectively, the tensor, the vector and the axial-vector parts of torsion and energy, momentum and angular momentum in the ASSF. We find the vector parts are in the radial and $\hat{e}_\theta$ directions, the axial-vector, momentum and angular momentum vanish identically, but the energy distribution is different from zero. The vanishing axial-vector part of torsion gives us the result that there occurs no deviation in the spherical symmetry of the spacetime. Consequently, there exists no inertia field with respect to a Dirac particle, and the spin vector of a Dirac particle becomes constant. The result for the energy is the same as obtained by Radinschi. Next, this work also (a) supports the viewpoint of Lessner that the Møller energy-momentum complex is a powerful concept for the energy-momentum, (b) sustains the importance of the energy-momentum definitions in the evaluation of the energy distribution of a given spacetime, and (c) supports the hypothesis by Cooperstock that the energy is confined to the region of non-vanishing energy-momentum tensor of the matter and all non-gravitational fields.

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1 Introduction

Scalar fields are predicted as a fundamental interaction in Kaluza–Klein and superstring theories. The scalar fields are fundamental components of the Brans–Dicke theory and of the inflationary model. Further, the scalar fields are a good nominee for dark matter in spiral galaxies. Since scalar fields interact very weakly with matter, we have never seen one, but many of the theories containing scalar fields are in good concordance with measurements in weak gravitational fields. We suppose that scalar fields can play an important role in strong gravitational fields like at the origin of the universe or in pulsars or black holes.

The axially symmetric scalar solution to the field equations derived from the action for minimal coupling to a scalar field is [1, 2]

$$\mathrm{d}s^2 = \left(1-\frac{2m}{r}\right)\mathrm{d}t^2 - \mathrm{e}^{2k_a}\left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2\left(\mathrm{e}^{2k_a}\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\theta^2\right), \tag{1}$$

with

$$\mathrm{e}^{-2a^2k_a} = 1 + \frac{m^2\sin^2\theta}{r^2\left(1-\frac{2m}{r}\right)}, \tag{2}$$

$$\varphi = \frac{1}{2a}\ln\left(1-\frac{2m}{r}\right), \tag{3}$$

where a is a constant of integration and φ is the scalar field. The solution given above is one of the new classes of solutions to the Einstein–Maxwell theory non-minimally coupled to a dilatonic field [2]. The line-element given by (1) is almost spherically symmetric and represents a gravitational monopole with scalar field. In addition, the scalar field deforms the spherically symmetry. We see that when $a \to \infty$ we recover the Schwarzschild solution.

The name of teleparallel gravitation theory is normally used to describe the general three parameter theory introduced in [3]. The teleparallel equivalent of Einstein's theory of general relativity [4] can be understood as a gauge theory for the translation group based on Weitzenböck geometry [5]. In the teleparallel theory, the gravitational interaction is described by a force similar to the Lorentz force equation of electrodynamics, with torsion playing the role of force and with identically vanishing curvature tensor [6]. The basic entity of teleparallel theory is the non-trivial tetrad field $h^a{}_\mu$, while in general relativity the metric tensor plays the role of the basic entity. In spite of these fundamental differences, the two theories provide equivalent descriptions of the gravitational interaction [7, 9]. This shows that torsion and curvature might be simply two alternative ways of describing the gravitational field and also sustains the fact that the symmetric energy-momentum tensor is a source in both gravitation theories; i.e., the source of curvature in general relativity and the source of torsion in teleparallel theory. In some other theories [4, 8], torsion is only relevant when spins are important [7, 9]. Hence, this point of view indicates that torsion might represent additional degrees of freedom as compared to curvature.

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It is worth to mention that the tetrad formalism itself has some advantages which come mainly from its independence from the equivalence principle and the consequent suitability for the discussion of quantum issues. Furthermore, according to the teleparallel equivalent of general relativity, curvature and torsion are alternative ways of describing the gravitational field and consequently are related to the same degrees of freedom of gravity; however, more general gravity theories [9], like for example Einstein–Cartan and gauge theories for the Poincaré and the affine groups [10], consider curvature and torsion as representing independent degrees of freedom. In these models, differently from teleparallel gravity, torsion becomes relevant only when spins are important [7,9]. According to this point of view, torsion represents additional degrees of freedom in relation to curvature, and consequently new physics phenomena might be associated with it [11,12].

An important point of the teleparallel equivalent of general relativity is that it allows for the definition of an energy-momentum gauge current j_i^μ for the gravitational field which is covariant under a general spacetime coordinate transformation and transforms covariantly under a global tangent-space Lorentz transformation [12]. This means essentially that j_i^μ is a true spacetime tensor but not a tangent-space tensor. Then, by rewriting the gauge field equation in a purely spacetime form, it becomes Einstein's equation, and the gauge current j_i^μ reduces to the canonical energy-momentum pseudotensor of the gravitational field. Teleparallel gravity, therefore, seems to provide a more appropriate environment to deal with the energy problem since in the ordinary context of general relativity, the energy-momentum density for the gravitational field will always be represented by a pseudotensor [13].

In this work, we obtain the torsion fields and energy-momentum distribution (due to matter plus fields including gravity) associated with an ASSF. In the next section, we obtain the tensor, vector and the axial-vector parts of the tensor in an ASSF. In Sect. 3, we calculate the energy-momentum in an ASSF using the teleparallel gravity version of Møller's energy-momentum definition. Finally, Sect. 4 is devoted to discussions.

Our notation and conventions entail $c = h = 1$, the signature of the metric being $(+,-,-,-)$.

2 Axial-vector torsion

We will use the Greek alphabet $(\mu,\nu,\alpha,\beta,... = 0,1,2,3)$ to denote tensor indices related to spacetime. The Latin alphabet $(a,b,c,d,... = 0,1,2,3)$ will be used to denote local Lorentz (or tangent-space) indices, whose associated metric is

$$\eta_{ij} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{4}$$

Tensor and Lorentz indices can be changed into each other with the use of a tetrad field $h^a{}_\mu$ which satisfies

$$h^i{}_\alpha h_i{}^\beta = \delta^\beta_\alpha , \quad h^i{}_\alpha h_j{}^\alpha = \delta^i_j . \tag{5}$$

A non-trivial field can be used to describe the linear Weitzenböck connection:

$$\Gamma^\lambda{}_{\alpha\beta} = h_i{}^\lambda \partial_\beta h^i{}_\alpha , \tag{6}$$

with respect to which the tetrad is parallel:

$$\nabla_\lambda h^i{}_\rho \equiv \partial_\lambda h^i{}_\rho - \Gamma^\sigma{}_{\lambda\rho} h^i{}_\sigma = 0 . \tag{7}$$

We also have the following relation for the Weitzenböck connection:

$$\Gamma^\lambda{}_{\alpha\beta} = \widetilde{\Gamma}^\lambda_{\alpha\beta} - \Upsilon^\lambda_{\alpha\beta} , \tag{8}$$

where $\widetilde{\Gamma}^\lambda_{\alpha\beta}$ is the Levi-Civita connection of the metric $g_{\alpha\beta} = \eta_{ij} h^i{}_\alpha h^j{}_\beta$, which is given by

$$\widetilde{\Gamma}^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu} \right) , \tag{9}$$

and

$$\Upsilon^\lambda_{\alpha\beta} = \frac{1}{2} \left(T_{\alpha\ \beta}{}^\lambda + T_{\beta\ \alpha}{}^\lambda - T^\lambda{}_{\alpha\beta} \right) \tag{10}$$

defines the contorsion tensor, with

$$T^\lambda{}_{\alpha\beta} = \Gamma^\lambda{}_{\beta\alpha} - \Gamma^\lambda{}_{\alpha\beta} , \tag{11}$$

which is the torsion of the Weitzenböck connection [14].

The torsion tensor can be decomposed into three irreducible parts under the group of global Lorentz transformation [3]: the tensor part

$$t_{\alpha\mu\nu} = \frac{1}{2} \left(T_{\alpha\mu\nu} + T_{\mu\alpha\nu} \right) + \frac{1}{6} \left(g_{\nu\alpha} V_\mu + g_{\nu\mu} V_\alpha \right) - \frac{1}{3} g_{\alpha\mu} V_\nu , \tag{12}$$

the vector part

$$V_\mu = T^\alpha{}_{\alpha\mu} , \tag{13}$$

and the axial-vector part

$$A^\mu = \frac{1}{6} \varepsilon^{\mu\nu\alpha\beta} T_{\nu\alpha\beta} . \tag{14}$$

The torsion tensor can now be expressed in terms of these irreducible components as follows:

$$T_{\alpha\mu\nu} = \frac{1}{2} \left(t_{\alpha\mu\nu} - t_{\alpha\nu\mu} \right) + \frac{1}{3} \left(g_{\alpha\mu} V_\nu - g_{\alpha\nu} V_\mu \right) + \varepsilon_{\alpha\mu\nu\sigma} A^\sigma , \tag{15}$$

where

$$\varepsilon^{\alpha\mu\nu\sigma} = \frac{1}{\sqrt{-g}} \delta^{\alpha\mu\nu\sigma} . \tag{16}$$

Here $\delta = \delta^{\alpha\mu\nu\sigma}$ and $\overline{\delta} = \delta_{\alpha\mu\nu\sigma}$ are completely skew symmetric tensor densities of weight -1 and $+1$, respectively [3]. In is worth mentioning here that the deviation of axial symmetry from the spherical symmetry is represented by the axial-vector torsion. It has been shown, in both general relativity and teleparallel theory, that the spin precession of a Dirac particle in torsion gravity is related to the torsion axial-vector [15] by

$$\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}t} = -\frac{3}{2}\mathbf{A}\times\mathbf{S}\,, \tag{17}$$

where $\mathbf{S}$ is the spin vector of a Dirac particle and $\mathbf{A}$ is the space-like part of the torsion axial-vector. The Hamiltonian would be of the form

$$\delta H = -\frac{3}{2}\mathbf{A}\cdot\boldsymbol{\sigma}\,, \tag{18}$$

where $\boldsymbol{\sigma}$ is the particle spin [16].

The general form of the tetrad h_i^μ having spherical symmetry was given by Robertson [17]. In cartesian form it can be written as

$$\begin{aligned} &h_0^0 = \mathrm{i}W\,, \quad h_a^0 = Zx^a\,, \quad h_0^\alpha = \mathrm{i}Hx^\alpha\,,\\ &h_a^\alpha = K\delta_a^\alpha + Sx^a x^\alpha + \epsilon_{a\alpha\beta}Gx^\beta\,, \end{aligned} \tag{19}$$

where W, K, Z, H, S, and G are functions of t and $r = \sqrt{x^a x^a}$, and the zeroth vector h_0^μ has the factor $\mathrm{i}^2 = -1$ to preserve the Lorentz signature, and we have the tetrad of Minkowski spacetime, $h_a^\mu = \mathrm{diag}\,(\mathrm{i}, \delta_a^\alpha)$, where $a = 1, 2, 3$. Using the general coordinate transformation we have

$$h_{a\mu} = \frac{\partial \mathbf{X}^{\nu'}}{\partial \mathbf{X}^\mu} h_{a\nu}\,, \tag{20}$$

where $\{\mathbf{X}^\mu\}$ and $\{\mathbf{X}^{\nu'}\}$ are, respectively, the isotropic and Schwarzschild coordinates (t, r, θ, ϕ). In the spherical, static and isotropic coordinate system

$$\mathbf{X}^1 = r\sin\theta\cos\phi\,, \tag{21}$$
$$\mathbf{X}^2 = r\sin\theta\sin\phi\,, \tag{22}$$
$$\mathbf{X}^3 = r\cos\theta\,, \tag{23}$$

we obtain the tetrad components of h_a^μ as follows:

$$\begin{aligned} h_a^\mu = {} & \frac{\mathrm{i}\delta_a^0\delta_0^\mu}{\sqrt{1-\frac{2m}{r}}} \\ & + \mathrm{e}^{-k_a}\sqrt{1-\frac{2m}{r}}\left(s\theta c\phi\delta_a^1\delta_1^\mu + s\theta s\phi\delta_a^2\delta_1^\mu + c\theta\delta_a^3\delta_1^\mu\right) \\ & + \frac{\mathrm{e}^{-k_a}}{r}\left(c\theta c\phi\delta_a^1\delta_2^\mu + c\theta c\phi\delta_a^2\delta_2^\mu - s\theta\delta_a^3\delta_2^\mu\right) \\ & - \frac{1}{r}\left(\frac{s\phi}{s\theta}\delta_a^1\delta_3^\mu - \frac{c\phi}{s\theta}\delta_a^2\delta_3^\mu\right). \end{aligned} \tag{24}$$

Here, we have introduced the following notation: $s\theta = \sin\theta, c\theta = \cos\theta, s\phi = \sin\phi$ and $c\phi = \cos\phi$. For an axially symmetric scalar field, the matrix of the $g_{\mu\nu}$ is given by

$$\begin{aligned} g_{\mu\nu} = {} & \left(1-\frac{2m}{r}\right)\delta_\mu^0\delta_\nu^0 - \mathrm{e}^{2k_a}\left(1-\frac{2m}{r}\right)^{-1}\delta_\mu^1\delta_\nu^1 \\ & - r^2\mathrm{e}^{2k_a}\delta_\mu^2\delta_\nu^2 - r^2\sin^2\theta\delta_\mu^3\delta_\nu^3\,, \end{aligned} \tag{25}$$

and its inverse matrix $g^{\mu\nu}$ is

$$\begin{aligned} g^{\mu\nu} = {} & \left(1-\frac{2m}{r}\right)^{-1}\delta_0^\mu\delta_0^\nu - \mathrm{e}^{-2k_a}\left(1-\frac{2m}{r}\right)\delta_1^\mu\delta_1^\nu \\ & - \frac{\mathrm{e}^{-2k_a}}{r^2}\delta_2^\mu\delta_2^\nu - \frac{1}{r^2\sin^2\theta}\delta_3^\mu\delta_3^\nu\,. \end{aligned} \tag{26}$$

The corresponding non-vanishing components of the contorsion tensor are

$$T^0{}_{01} = \frac{m}{2mr - r^2}\,, \qquad T^1{}_{12} = -\partial_\theta k_a\,, \tag{27}$$
$$T^2{}_{12} = \frac{1}{r}\left[1-\left(1-\frac{2m}{r}\right)^{-1/2}\right] + \partial_r k_a\,, \tag{28}$$
$$T^3{}_{13} = \frac{1}{r}\left[1-\mathrm{e}^{k_a}\left(1-\frac{2m}{r}\right)^{-1/2}\right], \tag{29}$$
$$T^3{}_{23} = \left(1-\mathrm{e}^{k_a}\right)\cot\theta\,. \tag{30}$$

Hence, the non-vanishing components of the vector part of the torsion turn out to be

$$\begin{aligned} V_1(r,\theta) = {} & \frac{1}{(2m-r)}\left[2-(1+\mathrm{e}^{k_a})\left(1-\frac{2m}{r}\right)^{1/2}\right] \\ & - \frac{3m - r(r-2m)\partial_r k_a}{(2m-r)r}\,, \end{aligned} \tag{31}$$
$$V_2(r,\theta) = \left(\mathrm{e}^{k_a}-1\right)\cot\theta - \partial_\theta k_a\,. \tag{32}$$

In view of these results, the axial-vector part of torsion vanishes, i.e.,

$$A^\mu(r,\theta) = 0\,. \tag{33}$$

Therefore, in space-like vector form, the axial-vector becomes

$$\mathbf{A}(r,\theta) = 0\,. \tag{34}$$

From this point of view, the spin vector of the Dirac particle is constant, and the corresponding Hamiltonian induced by the axial-vector spin coupling vanishes. Since the torsion plays the role of the gravitational force in teleparallel gravity, a spinless particle will obey the force equation [14, 18, 19] in the gravitational field:

$$\frac{\mathrm{d}u_\lambda}{\mathrm{d}s} - \Gamma_{\mu\lambda\nu}u^\mu u^\nu = T_{\mu\lambda\nu}u^\mu u^\nu\,. \tag{35}$$

The left hand side of this equation is the Weitzenböck covariant derivative of u_λ along the world line of the particle. The presence of the torsion tensor on its right hand side means essentially that torsion plays the role of an external force in teleparallel theory of gravity.

3 Energy-momentum distribution

After the pioneering expression by Einstein [20, 21] for the energy and momentum distributions of the gravitational field, many attempts have been proposed to resolve the gravitational energy problem [22–28]. Except for the definition of Møller, these definitions give meaningful results only if the calculations are performed in *cartesian* coordinates. Møller constructed an expression which enables one to evaluate energy and momentum in any coordinate system. Next, Lessner [29] argued that the Møller prescription is a powerful concept for energy-momentum in general relativity.

In general relativity, Virbhadra [30], using the energy and momentum complexes of Einstein, Landau–Lifshitz, Papapetrou and Weinberg for a general non-static spherically symmetric metric of the Kerr–Schild class, showed that all of these energy-momentum formulations give the same energy distribution as in the Penrose energy-momentum formulation. Several examples of particular spacetimes have been investigated and different energy-momentum pseudotensors are known to give the same energy distribution for a given spacetime [31–76].

Recently, the problem of energy-momentum localization has also been considered in teleparallel gravity [77]. Møller showed that a tetrad description of a gravitational field equation allows for a more satisfactory treatment of the energy-momentum complex than does general relativity. Therefore, we have applied the super-potential method by Mikhail et al. [78] to calculate the energy-momentum of the central gravitating body. Vargas [77], using the definitions of Einstein and Landau–Lifshitz in teleparallel gravity, found that the total energy is zero in Friedmann–Robertson–Walker spacetimes.

Considerable efforts have also been made in constructing super-energy tensors [79]. Motivated by the works of Bel [80–82] and independently of Robinson [83], many investigations have been carried out in this field [84–87].

In the literature, Radinschi [88] considered the Møller energy-momentum complex in general relativity for an axially symmetric scalar field to calculate the energy distribution for a given spacetime and found that the energy distribution is given by the mass m;

$$E(r) = m\,. \tag{36}$$

It is really interesting to evaluate the teleparallel gravitational energy distribution in this spacetime model.

The super-potential of Møller's in teleparallel gravity is given by Mikhail et al. [78] as

$$\begin{aligned}\varDelta_\mu^{\nu\beta} = \frac{(-g)^{1/2}}{2\kappa} &\left(\delta_\chi^\tau\left[\delta_\rho^\nu\delta_\sigma^\beta - \delta_\sigma^\nu\delta_\rho^\beta\right] + \delta_\rho^\tau\left[\delta_\sigma^\nu\delta_\chi^\beta - \delta_\chi^\nu\delta_\sigma^\beta\right]\right.\\ &\left.-\delta_\sigma^\tau\left[\delta_\chi^\nu\delta_\rho^\beta - \delta_\rho^\nu\delta_\chi^\beta\right]\right)\\ &\times\left\{\Phi^\rho g^{\sigma\chi} g_{\mu\tau} - \lambda g_{\tau\mu}\xi^{\chi\rho\sigma} - (1-2\lambda) g_{\tau\mu}\xi^{\sigma\rho\chi}\right\};\end{aligned} \tag{37}$$

here $\xi_{\alpha\beta\mu}$ is the contorsion tensor given by

$$\xi_{\alpha\beta\mu} = h_{i\alpha} h^i{}_{\beta;\mu}\,, \tag{38}$$

where the semicolon denotes covariant differentiation with respect to the Christoffel symbols

$$\left\{\begin{matrix}\alpha\\ \mu\nu\end{matrix}\right\} = \frac{1}{2} g^{\alpha\beta}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right) \tag{39}$$

and Φ_μ is the basic vector field defined by

$$\Phi_\mu = \xi^\rho_{\mu\rho}\,; \tag{40}$$

κ is the Einstein constant, and λ is a free dimensionless parameter.

In a spacetime with absolute parallelism the teleparallel vector fields $h_i{}^\mu$ define the non-symmetric connection

$$\Gamma^\alpha{}_{\mu\beta} = h_i{}^\alpha \partial_\beta h^i{}_\mu\,. \tag{41}$$

The curvature tensor which is defined by $\Gamma^\alpha{}_{\mu\beta}$ is identically vanishing. Møller constructed a gravitational theory based on this spacetime. In this gravitation theory the field variables are the 16 tetrad components $h_i{}^\mu$, from which the metric tensor is defined by

$$g^{\alpha\beta} = h^\alpha{}_i h^\beta{}_j \eta^{ij}\,. \tag{42}$$

We assume an imaginary value for the vector $h_0{}^\mu$ in order to have a Lorentz signature. We note that associated with any tetrad field $h_i{}^\mu$ there is a metric field defined uniquely by (42), while a given metric $g^{\alpha\beta}$ does not determine the tetrad field completely, for any local Lorentz transformation of the tetrads $h_i{}^\mu$ leads to a new set of tetrads which also satisfy (42).

The energy-momentum density is given by [89, 90]

$$\varXi_\alpha^\beta = \varDelta_{\alpha,\lambda}^{\beta\lambda}\,, \tag{43}$$

where the comma denotes ordinary differentation. The energy E and momentum components P_i are expressed by the volume integral [89, 90]

$$E = \lim_{r\to\infty}\int_{r=\text{constant}} \varXi_0^0 \mathrm{d}x\mathrm{d}y\mathrm{d}z\,, \tag{44}$$

$$P_i = \lim_{r\to\infty}\int_{r=\text{constant}} \varXi_i^0 \mathrm{d}x\mathrm{d}y\mathrm{d}z\,; \tag{45}$$

here the index i takes the values from 1 to 3. The angular momentum of a general relativistic system is given by [89, 90]

$$J_i = \lim_{r\to\infty}\int_{r=\text{constant}} \left(x_j \varXi_k^0 - x_k \varXi_j^0\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z\,, \tag{46}$$

where i, j and k cyclically take the values 1, 2 and 3.

After making the required calculations [91, 92], the non-vanishing component of Møller's super-potential $\varDelta_\mu^{\nu\beta}$ is obtained as

$$\varDelta_0^{01}(r,\theta) = \frac{2m}{\kappa}\sin\theta\,, \tag{47}$$

while the momentum density distributions take the form

$$\Xi_1^0(r,\theta)=0\,,\quad \Xi_2^0(r,\theta)=0\,,\quad \Xi_3^0(r,\theta)=0\,, \tag{48}$$

and the angular momentum of the system becomes

$$J_i(r,\theta)=0\,. \tag{49}$$

Therefore, we find the following energy:

$$E(r)=m\,, \tag{50}$$

and one can easily see that the momentum components are

$$\mathbf{P}(r)=0\,. \tag{51}$$

These are the same results as obtained in general relativity, and they are also independent of the teleparallel dimensionless coupling parameter, which means that this is valid not only in the teleparallel equivalent of general relativity but in any teleparallel model. The Møller energy in teleparallel gravity and general relativity sustains the viewpoint of Lessner that the Møller energy-momentum complex is a powerful concept of energy and momentum.

4 Discussions

This work is devoted to finding the teleparallel version of an axially symmetric scalar field and calculating the energy, momentum and angular momentum distributions associated with a given model. For this purpose, the tetrads of the spacetime solution is applied to the field equation of teleparallel gravity by using a coordinate transformation. Some classic solutions of Einstein's field equations have already been translated into the teleparallel gravity language. This study adds one more solution. It is always enriching to focus on known things from another point of view, so that the endeavor is in itself commendable.

For the ASSF, the vector parts are in the radial and $\hat{e}_\theta$ directions; the axial-vector torsion vanishes identically, and there occurs no deviation in the spherical symmetry of the spacetime. Consequently, there exists no inertia field with respect to a Dirac particle, and the spin vector of a Dirac particle becomes constant.

The energy distribution (due to matter and fields including gravity) was computed to be

$$E(r)=m\,, \tag{52}$$

while the momentum and angular momentum distributions vanishes. The energy calculated is exactly the same as calculated in Einstein's theory of general relativity by Radinschi. Furthermore, (a) the results obtained in teleparallel gravity are also independent of the teleparallel dimensionless coupling parameter, which means that it is valid not only in the teleparallel equivalent of general relativity, but in any teleparallel model; (b) this study supports the viewpoint of Lessner that the Møller energy-momentum complex is a powerful concept of energy and momentum; (c) it sustains the importance of the energy-momentum definitions in the evaluation of the energy distribution of a given spacetime; (d) it supports the hypothesis by Cooperstock [93] that the energy is confined to the region of non-vanishing energy-momentum tensor of matter and all non-gravitational fields.

Appendix : Tensor part of the torsion

In this section, we give the non-vanishing components of the tensor part of the torsion.

$$t_{001}=\frac{1}{3r}\left[2-\left(e^{2k_a}+1\right)\left(1-\frac{2m}{r}\right)^{1/2}\right]-\frac{6m-r(r-2m)\,(k_a)_{,r}}{3r^2}\,, \tag{A.1}$$

$$t_{002}=\frac{1}{3}\left(1-\frac{2m}{r}\right)\left[(k_a)_{,\theta}-\left(e^{2k_a}-1\right)\cot\theta\right]\,, \tag{A.2}$$

$$t_{010}=t_{100}=\frac{1}{6r}\left[\left(e^{2k_a}+1\right)\left(1-\frac{2m}{r}\right)^{1/2}-2\right]+\frac{6m-r(r-2m)(k_a)_{,r}}{6r^2}\,, \tag{A.3}$$

$$t_{020}=t_{200}=\frac{1}{6}\left(1-\frac{2m}{r}\right)\left[\left(e^{2k_a}-1\right)\cot\theta-(k_a)_{,\theta}\right]\,, \tag{A.4}$$

$$t_{112}=-\frac{e^{2k_a}r}{6m-3r}\left[\left(e^{2k_a}-1\right)\cot\theta+2(k_a)_{,\theta}\right]\,, \tag{A.5}$$

$$t_{121}=t_{211}=\frac{e^{2k_a}r}{12m-6r}\left[\left(e^{2k_a}-1\right)\cot\theta+2(k_a)_{,\theta}\right]\,, \tag{A.6}$$

$$t_{122}=t_{212}=-\frac{re^{2k_a}\left(1-\frac{2m}{r}\right)^{1/2}}{6(r-2m)}\times\left\{m.\left[4-2e^{k_a}-3\left(1-\frac{2m}{r}\right)^{1/2}\right]+\left[\left(1-\frac{2m}{r}\right)^{1/2}+e^{k_a}-2\right]+2\left(1-\frac{2m}{r}\right)^{1/2}r(r-2m)(k_a)_{,r}\right]\,, \tag{A.7}$$

$$t_{133}=t_{313}=\frac{r^2\sin^2\theta}{12m-6r}\left[1+\left(1-2e^{k_a}\right)\left(1-\frac{2m}{r}\right)^{1/2}\right]-\frac{r\sin^2\theta}{12m-6r}\left(3m-(2m-r)(k_a)_{,r}\right)\,, \tag{A.8}$$

$$t_{221}=\frac{re^{2k_a}\left(1-\frac{2m}{r}\right)^{1/2}}{3(r-2m)}\times\left\{m\left[4-2e^{k_a}-3\left(1-\frac{2m}{r}\right)^{1/2}\right]+\left[\left(1-\frac{2m}{r}\right)^{1/2}+e^{k_a}-2\right]\right.$$

$$+2\left(1-\frac{2m}{r}\right)^{1/2} r(r-2m)(k_a)_{,r}\Bigg\}\,, \qquad \text{(A.9)}$$

$$t_{233}=t_{323}=\frac{r^2\sin\theta}{6}\left[2\left(\mathrm{e}^{2k_a}-1\right)\cos\theta+(k_a)_{,\theta}\sin\theta\right]\,, \qquad \text{(A.10)}$$

$$t_{331}=\frac{r\sin^2\theta}{6m-3r}\Bigg\{3m-(2m-r)(k_a)_{,r} - \left[1+\left(1-2\mathrm{e}^{k_a}\right)\left(1-\frac{2m}{r}\right)^{1/2}\right]r\Bigg\}\,, \qquad \text{(A.11)}$$

$$t_{332}=\frac{r^2\sin\theta}{3}\left[2\left(\mathrm{e}^{2k_a}-1\right)\cos\theta+(k_a)_{,\theta}\sin\theta\right]\,. \qquad \text{(A.12)}$$

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